

Gravitational sources induced by exotic smoothness and fermions as knot complements

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Abstract In this paper we will discuss Brans conjecture that exotic smoothness serves as an additional gravitational source naturally arising from the handlebody construction of the exotic \mathbb{R}^4 . We will consider the two possible classes, the large and the small exotic \mathbb{R}^4 . Then we calculate the Einstein-Hilbert action for both exotic \mathbb{R}^4 to show the appearance of spinor fields. Then we discuss the physical properties of these spinor fields to relate them to fermions. Finally we identify the corresponding 3-manifolds as knot complements of hyperbolic knots, i.e. the knot complements are hyperbolic 3-manifolds with finite volume. With the help of this result we confirm the Brans conjecture for both kinds of exotic \mathbb{R}^4 . **Keywords:** exotic \mathbb{R}^4 , spinor field by exotic smoothness, fermions as knot complements, Brans conjecture

1 Introduction

The existence of exotic (non-standard) smoothness on topologically simple 4-manifolds such as exotic \mathbb{R}^4 or $S^3 \times \mathbb{R}$, has been known since the early eighties but the use of them in physical theories has been seriously hampered by the absence of finite coordinate presentations. However, the work of Bizaca and Gompf [BG96] provides a handle body representation of an exotic \mathbb{R}^4 which can serve as an infinite, but periodic, coordinate representation.

Thus we are looking for the decomposition of manifolds into small non-trivial, easily controlled objects (like handles). As an example consider the 2-torus $T^2 = S^1 \times S^1$ usually covered by at least 4 charts. However, it can

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be also decomposed using two 1-handles $D^1 \times D^1$ attached to the 0-handle $D^0 \times D^2 = D^2$ along their boundary $\partial D^2 = S^1$ via the boundary component of the 1-handle $\partial D^1 \times D^1 = S^0 \times D^1$, the disjoint union of two lines $S^0 \times D^1 = D^1 \sqcup D^1$. Finally one has to add a 2-handle $D^2 \times D^0$ to get the closed manifold T^2 . Every 1-handle can be covered by (at least) two charts and finally we recover the covering by 4 charts. Both pictures are equivalent but the handle picture has one simple advantage: it reduces the number of fundamental pieces of a manifold and of the transition maps. The gluing maps of the handles can be seen as a generalization of transition maps. Then the handle picture presents only the most important of these gluing or transition maps, omitting the trivial transition maps.

In this paper we will present such a coordinate representation, albeit infinite, of an exotic \mathbb{R}^4 based on the handle body decomposition of Bizaca and Gompf. We suggest that one of the consequences of this approach would be to suggest a positive answer for the Brans conjecture [AMB12], that exotic smoothness serves as an additional gravitational source as a spinor field naturally arising from the handlebody construction. The compact case was worked out in [AMR12]. This might be considered as a construction analogous to using the metric as a physical field once Einstein thought to look at gravity as a geometric effect. In other words, if we look for exotic smoothness effects in physics, the appearance of the spinor field in the periodic end construction parallels Einstein's looking to geometry as physics and then choosing the metric for gravity.

2 Construction of exotic \mathbb{R}^4

Our model of space-time is the non-compact space topological \mathbb{R}^4 . The results can be easily generalized for other cases such as $S^3 \times \mathbb{R}$. In this section we will give some information about the construction of exotic \mathbb{R}^4 . The existence of a smooth embedding $\mathbb{R}^4 \rightarrow S^4$ of the exotic \mathbb{R}^4 into the 4-sphere splits all exotic \mathbb{R}^4 into two classes, large (no embedding) or small.

2.1 Preliminaries: Slice and non-slice knots

At first we start with some definitions from knot theory. A (smooth) knot K is a smooth embedding $S^1 \rightarrow S^3$. In the following we assume every knot to be smooth. Secondly we exclude wilderness of knots, i.e the knot is equivalent to a polygon in \mathbb{R}^3 or S^3 (tame knot). Furthermore, the n -disk is denoted by D^n with $\partial D^n = S^{n-1}$.

Definition 1 Smoothly Slice Knot: A knot in $\partial D^4 = S^3$ is smoothly slice if there exists a two-disk D^2 smoothly embedded in D^4 such that the image of $\partial D^2 = S^1$ is K .

An example of a slice knot is the so-called Stevedore's Knot (in Rolfsen notation 6_1 , see Fig. 1).

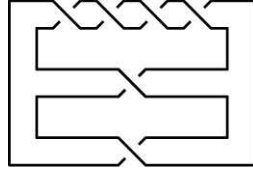


Fig. 1 a slice knot: Stevedore's knot 6_1

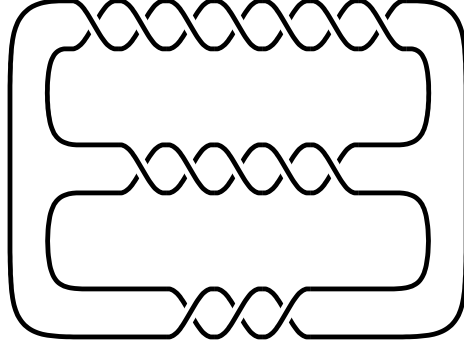


Fig. 2 topological, non-smoothly slice knot: pretzel knot $(-3, 5, 7)$

Definition 2 Flat Topological Embedding: Let X be a topological manifold of dimension n and Y a topological manifold of dimension m where $n < m$. A topological embedding $\rho : X \rightarrow Y$ is flat if it extends to a topological embedding $\rho : X \times D^{m-n} \rightarrow Y$.

Topologically Slice Knot: A knot K in ∂D^4 is topologically slice if there exists a two-disk D^2 flatly topologically embedded in D^4 such that the image of ∂D^2 is K .

Here we remark that the flatness condition is essential. Any knot $K \subset S^3$ is the boundary of a disc D^2 embedded in D^4 , which can be seen by taking the cone over the knot. But the vertex of the cone is a non-flat point (the knot is crashed to a point). The difference between the smooth and the flat topological embedding is the key for the following discussion. This innocent looking difference seem to imply that both definitions are equivalent. But deep results from 4-manifold topology gave a negative answer: there are topologically slice knots which are not smoothly slice. An example is the pretzel knot $(-3, 5, 7)$ (see Fig. 2).

In [Fre82a], Freedman gave a topological criteria for topological sliceness: the Alexander polynomial $\Delta_K(t)$ (the best known knot invariant, see [Rol76]) of the knot K has to be one, $\Delta_K(t) = 1$. An example how to measure the smooth sliceness is given by the smooth 4-genus $g_4(K)$ of the knot K , i.e. the minimal genus of a surface F smoothly embedded in D^4 with boundary $\partial F = K$ the knot. This surface F is called the *Seifert surface*. Therefore, if

the smooth 4-genus vanishes $g_4(K) = 0$ then the knot K bounds a 2-disk D^2 (surface of genus 0) given by the smooth embedding $D^2 \rightarrow D^4$ so that the image of $\partial D^2 \rightarrow \partial D^4$ is the knot K .

2.2 Large exotic \mathbb{R}^4 and non-slice knots

Large exotic \mathbb{R}^4 can be constructed by using the failure to arbitrarily split of a compact, simple-connected 4-manifold. For every topological 4-manifold one knows how to split this manifold *topologically* into simpler pieces using the work of Freedman [Fre82b]. But as shown by Donaldson [Don83], some of these 4-manifolds do not exist as smooth 4-manifolds. This contradiction between the continuous and the smooth case produces the first examples of exotic \mathbb{R}^4 [Gom83]. Unfortunately, the construction method is rather indirect and therefore useless for the calculation of the path integral contribution of the exotic \mathbb{R}^4 . But as pointed out by Gompf (see [Gom85] or [GS99] Exercise 9.4.23 on p. 377ff and its solution on p. 522ff), large exotic \mathbb{R}^4 can be also constructed by using smoothly non-slice but topologically slice knots. Especially one obtains an explicit construction which will be used in the calculations later.

Let K be a knot in ∂D^4 and X_K the two-handlebody obtained by attaching a two-handle to D^4 along K with framing 0. That means: one has a two-handle $D^2 \times D^2$ which is glued to the 0-handle D^4 along its boundary using a map $f : \partial D^2 \times D^2 \rightarrow \partial D^4$ so that $f(\cdot, x) = K \times x \subset S^3 = \partial D^4$ for all $x \in D^2$ (or the image $\text{im}(f) = K \times D^2$ is the solid knotted torus). Let $\rho : X_K \rightarrow \mathbb{R}^4$ be a flat topological embedding (K is topologically slice). For K a smoothly non-slice knot, the open 4-manifold

$$R^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \quad (1)$$

where $\text{int}\rho(X_K)$ is the interior of $\rho(X_K)$, is homeomorphic but non-diffeomorphic to \mathbb{R}^4 with the standard smoothness structure (both pieces are glued along the common boundary ∂X_K). The proof of this fact (R^4 is exotic) is given by contradiction, i.e. let us assume R^4 is diffeomorphic to \mathbb{R}^4 . Thus, there exists a diffeomorphism $R^4 \rightarrow \mathbb{R}^4$. The restriction of this diffeomorphism to X_K is a smooth embedding $X_K \hookrightarrow \mathbb{R}^4$. However, such a smooth embedding exists if and only if K is smoothly slice (see [GS99]). But, by hypothesis, K is not smoothly slice. Thus by contradiction, there exists a no diffeomorphism $R^4 \rightarrow \mathbb{R}^4$ and R^4 is exotic, homeomorphic but not diffeomorphic to \mathbb{R}^4 . Finally, we have to prove that R^4 is large. X_K , by construction, is compact and a smooth submanifold of R^4 . By hypothesis, K is not smoothly slice and therefore X_K can not smoothly embed in \mathbb{R}^4 . By restriction, $D^4 \subset X_K$ and also $\partial D^4 = S^3$ can not smoothly embed and therefore R^4 is a large exotic \mathbb{R}^4 .

2.3 Small exotic \mathbb{R}^4 and Casson handles

Small exotic \mathbb{R}^4 's are again the result of anomalous smoothness in 4-dimensional topology but of a different kind than for large exotic \mathbb{R}^4 's. In 4-manifold topology [Fre82b], a homotopy-equivalence between two compact, closed, simply-connected 4-manifolds implies a homeomorphism between them (a so-called h cobordism). But Donaldson [Don87] provided the first smooth counterexample, i.e. both manifolds are generally not diffeomorphic to each other. The failure can be localized in some contractible submanifold (Akbulut cork) so that an open neighborhood of this submanifold is a small exotic \mathbb{R}^4 . The whole procedure implies that this exotic \mathbb{R}^4 can be embedded in the 4-sphere S^4 .

The idea of the construction is simply given by the fact that every such smooth h-cobordism between non-diffeomorphic 4-manifolds can be written as a product cobordism except for a compact contractible sub-h-cobordism V , the Akbulut cork. An open subset $U \subset V$ homeomorphic to $[0, 1] \times \mathbb{R}^4$ is the corresponding sub-h-cobordism between two exotic \mathbb{R}^4 's. These exotic \mathbb{R}^4 's are called ribbon \mathbb{R}^4 's. They have the important property of being diffeomorphic to open subsets of the standard \mathbb{R}^4 . To be more precise, consider a pair (X_+, X_-) of homeomorphic, smooth, closed, simply-connected 4-manifolds.

Theorem 1 *Let W be a smooth h-cobordism between closed, simply connected 4-manifolds X_- and X_+ . Then there is an open subset $U \subset W$ homeomorphic to $[0, 1] \times \mathbb{R}^4$ with a compact subset $C \subset U$ such that the pair $(W \setminus C, U \setminus C)$ is diffeomorphic to a product $[0, 1] \times (X_- \setminus C, U \cap X_- \setminus C)$. The subsets $R_{\pm} = U \cap X_{\pm}$ (homeomorphic to \mathbb{R}^4) are diffeomorphic to open subsets of \mathbb{R}^4 . If X_- and X_+ are not diffeomorphic, then there is no smooth 4-ball in R_{\pm} containing the compact set $Y_{\pm} = C \cap R_{\pm}$, so both R_{\pm} are exotic \mathbb{R}^4 's.*

Thus, remove a certain contractible, smooth, compact 4-manifold $Y_- \subset X_-$ (called an Akbulut cork) from X_- , and re-glue it by an involution of ∂Y_- , i.e. a diffeomorphism $\tau : \partial Y_- \rightarrow \partial Y_-$ with $\tau \circ \tau = Id$ and $\tau(p) \neq \pm p$ for all $p \in \partial Y_-$. This argument was modified above so that it works for a contractible open subset $R_- \subset X_-$ with similar properties, such that R_- will be an exotic \mathbb{R}^4 if X_+ is not diffeomorphic to X_- . Furthermore R_- lies in a compact set, i.e. a 4-sphere or R_- is a small exotic \mathbb{R}^4 . In [DF92] Freedman and DeMichelis constructed also a continuous family of small exotic \mathbb{R}^4 .

Now we are ready to discuss the decomposition of a small exotic \mathbb{R}^4 by Bizaca and Gompf [BG96] by using special pieces, the handles forming a handle body. Every 4-manifold can be decomposed (seen as handle body) using standard pieces such as $D^k \times D^{4-k}$, the so-called k -handle attached along $\partial D^k \times D^{4-k}$ to the boundary $S^3 = \partial D^4$ of a 0-handle $D^0 \times D^4 = D^4$. The construction of the handle body can be divided into two parts. The first part is known as the Akbulut cork, a contractible 4-manifold with boundary a homology 3-sphere (a 3-manifold with the same homology as the 3-sphere). The Akbulut cork A_{cork} is given by a linking between a 1-handle and a 2-handle of framing 0. The second part is the Casson handle CH which will be considered now.

Let us start with the basic construction of the Casson handle CH . Let M be a smooth, compact, simple-connected 4-manifold and $f : D^2 \rightarrow M$ a (codimension-2) mapping. By using diffeomorphisms of D^2 and M , one can deform the mapping f to get an immersion (i.e. injective differential) generically with only double points (i.e. $\#|f^{-1}(f(x))| = 2$) as singularities [GG73]. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle $D^2 \times D^2$ induced by $f \times id : D^2 \times D^2 \rightarrow M$ from f . Then every double point (or self-intersection) of $f(D^2)$ leads to self-plumbings of the 2-handle $D^2 \times D^2$. A self-plumbing is an identification of $D_0^2 \times D^2$ with $D_1^2 \times D^2$ where $D_0^2, D_1^2 \subset D^2$ are disjoint sub-disks of the first factor disk¹. Consider the pair $(D^2 \times D^2, \partial D^2 \times D^2)$ and produce finitely many self-plumbings away from the attaching region $\partial D^2 \times D^2$ to get a kinky handle $(k, \partial^- k)$ where $\partial^- k$ denotes the attaching region of the kinky handle. A kinky handle $(k, \partial^- k)$ is a one-stage tower $(T_1, \partial^- T_1)$ and an $(n+1)$ -stage tower $(T_{n+1}, \partial^- T_{n+1})$ is an n -stage tower union kinky handles $\bigcup_{\ell=1}^n (T_\ell, \partial^- T_\ell)$ where two towers are attached along $\partial^- T_\ell$. Let T_n^- be $(\text{interior } T_n) \cup \partial^- T_n$ and the Casson handle

$$CH = \bigcup_{\ell=0} T_\ell^-$$

is the union of towers (with direct limit topology induced from the inclusions $T_n \hookrightarrow T_{n+1}$).

The main idea of the construction above is very simple: an immersed disk (disk with self-intersections) can be deformed into an embedded disk (disk without self-intersections) by sliding one part of the disk along another (embedded) disk to kill the self-intersections. Unfortunately the other disk can be immersed only. But the immersion can be deformed to an embedding by a disk again etc. In the limit of this process one "shifts the self-intersections into infinity" and obtains² the standard open 2-handle $(D^2 \times \mathbb{R}^2, \partial D^2 \times \mathbb{R}^2)$.

A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point $*$, having all edge paths infinitely extendable away from $*$. Each edge should be given a label $+$ or $-$. Here is the construction: $\text{tree} \rightarrow CH$. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. Each building block of a Casson handle, the "kinky" handle with n kinks³, is diffeomorphic to the n -times boundary-connected sum $\natural_n(S^1 \times D^3)$ (see appendix A) with two

¹ In complex coordinates the plumbing may be written as $(z, w) \mapsto (w, z)$ or $(z, w) \mapsto (\bar{w}, \bar{z})$ creating either a positive or negative (respectively) double point on the disk $D^2 \times 0$ (the core).

² In the proof of Freedman [Fre82b], the main complications come from the lack of control about this process.

³ The number of end-connected sums is exactly the number of self intersections of the immersed two handle.

attaching regions. Technically speaking, one region is a tubular neighborhood of band sums of Whitehead links connected with the previous block. The other region is a disjoint union of the standard open subsets $S^1 \times D^2$ in $\#_n S^1 \times S^2 = \partial(\natural_n S^1 \times D^3)$ (this is connected with the next block).

2.4 The Einstein-Hilbert action

In this section we will discuss the Einstein-Hilbert action functional

$$S_{EH}(M) = \int_M R\sqrt{g} d^4x \quad (2)$$

of the 4-manifold M and fix the Ricci-flat metric g as solution of the vacuum field equations of the exotic 4-manifold. The main part of our argumentation is additional contribution to the action functional coming from exotic smoothness.

In case of the large exotic \mathbb{R}^4 , we consider the decomposition

$$R^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \quad (3)$$

where R^4 is the large exotic \mathbb{R}^4 . For the parts of the decomposition we obtain the action functionals

$$\begin{aligned} S_{EH}(\mathbb{R}^4 \setminus \text{int}\rho(X_K)) &= \int_{\mathbb{R}^4 \setminus \text{int}\rho(X_K)} R\sqrt{g} d^4x + \int_{\partial X_K} H\sqrt{h} d^3x \\ S_{EH}(X_K) &= \int_{X_K} R\sqrt{g} d^4x - \int_{\partial X_K} H\sqrt{h} d^3x \end{aligned}$$

including the contribution of the boundary ∂X_K with respect to different orientations and H is the trace of the second fundamental form (mean curvature) of the boundary in the metric g .

As explained above, a small exotic \mathbb{R}^4 can be decomposed into a compact subset A_{cork} (Akbulut cork) and a Casson handle (see [BG96]). Especially this exotic \mathbb{R}^4 depends strongly on the Casson handle, i.e. non-diffeomorphic Casson handles lead to non-diffeomorphic \mathbb{R}^4 's. Thus we have to understand the analytical properties of a Casson handle. In [Kat04], the analytical properties of the Casson handle were discussed. The main idea is the usage of the theory of end-periodic manifolds, i.e. an infinite periodic structure generated by W glued along a compact set A_{cork} to get for the interior

$$\mathbb{R}_\theta^4 = \text{int}(A_{cork} \cup_N W \cup_N W \cup_N \dots)$$

the end-periodic manifold. The definition of an end-periodic manifold is very formal (see [Tau87]) and we omit it here. All Casson handles generated by a balanced tree have the structure of end-periodic manifolds as shown in [Kat04]. By using the theory of Taubes [Tau87] one can construct a metric on $\dots \cup_N$

$W \cup_N W \cup_N \dots$ by using the metric on W . Then a metric g in \mathbb{R}_θ^4 transforms to a periodic function \hat{g} on the infinite periodic manifold

$$\tilde{Y} = \dots \cup_N W_{-1} \cup_N W_0 \cup_N W_1 \cup_N \dots$$

where W_i is the building block W at the i th place. Then the action of \mathbb{R}_θ^4 can be divided into many parts

$$\begin{aligned} S_{EH}(A_{cork}) &= \int_{A_{cork}} R\sqrt{g}d^4x - \int_{\partial A_{cork}} H\sqrt{h}d^3x \\ S_{EH}(W_i) &= \int_{W_i} R\sqrt{g}d^4x + \int_N H\sqrt{h}d^3x \end{aligned}$$

again including the boundaries $N = \partial W$ and ∂A_{cork} . In any case we can reduce the problem to the discussion of the action

$$S_{EH}(\Sigma) = \int_{\Sigma} H\sqrt{h}d^3x \quad (4)$$

along the boundary Σ (a 3-manifold). It is a surprise that this integral agrees with the Dirac action of a spinor describing the (immersed) boundary, see below.

3 Immersed surfaces and the Dirac action

In the following we will show that the action (4) is completely determined by the knotted torus $\partial N(K) = K \times S^1$ and its mean curvature $H_{\partial N(K)}$. This knotted torus is an immersion of a torus $S^1 \times S^1$ into \mathbb{R}^3 . The well-known *Weierstrass representation* can be used to describe this immersion. As proved in [KS96, Fri98] there is an equivalent representation via spinors. This so-called *Spin representation* of a surface gives back an expression for $H_{\partial N(K)}$ and the Dirac equation as geometric condition on the immersion of the surface. As we will show below, the term (4) can be interpreted as Dirac action of a spinor field.

3.1 From 3-manifolds to immersed surfaces

The action (4) depends on the 3-manifold Σ as the boundary of an appropriated 4-manifold M . Then the embedding of this boundary depends on the 3-manifold Σ which we have to describe first. The relation between the 4-manifold and the boundary (a 3-manifold) is very close. In particular, the 4-manifolds in this paper can be obtained by adding 2-handles (glued to the 0-handle by using knots). Then one can construct the 3-manifold by similar methods. The core of this method is the following result: Let Σ be an arbitrary 3-manifold and S^3 the 3-sphere. Cut out a solid torus $T = S^1 \times D^2$ from

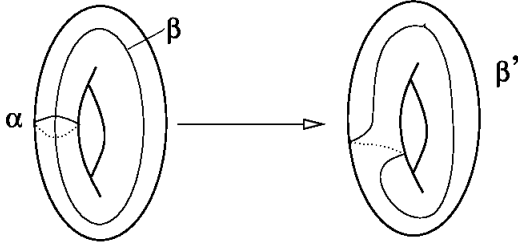


Fig. 3 Dehn twist

both manifolds then $\Sigma \setminus T$ and $S^3 \setminus T$ are homeomorphic (and also diffeomorphic). So, every 3-manifold can be generated by a procedure (called surgery): cutting out solid tori from the 3-sphere S^3 and then pasting them back in, but along different homeomorphisms of their boundaries. Then the homeomorphisms of the boundaries, the usual torus $T^2 = S^1 \times S^1$, determine the 3-manifold completely. Homeomorphisms of the torus T^2 are well understood using Dehn twists. In a Dehn twist, one cut the torus to obtain a cylinder and past both ends together after a full twist of one end (see Fig. 3).

Equally one can also do a twist along the other curve α . For a coordinate description of this procedure one considers the torus as a product of two circles $S^1 \times S^1$ (denoted by α, β in the Fig.). Let (θ, ϕ) be two angle coordinates (range $[0, 2\pi)$, for each S^1 factor. If $u(\alpha)$ is a smooth function equal to one near π but zero elsewhere, we represent the (p, q) twist by

$$(\theta, \phi) \rightarrow (\theta + p2\pi u(\phi), \phi + q2\pi u(\theta)).$$

Or, we can define the $(p, q) \in \mathbb{Z}^2$ twisted torus as identification space resulting from identifying $(x, y) \sim (m\pi x, n\pi y)$ for any $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$. By a Dehn twist, one obtains a knotting of the torus. But more importantly we obtain a surface inside of the 3-manifold (unique up to diffeomorphisms) so that the embedding of the 3-manifold can be described by the embedding of this surface. Therefore, two 3-manifolds agree on the complement of some disjoint solid tori, i.e. we have for a 3-manifold $\Sigma = (S^3 \setminus (D^2 \times S^1)) \cup (K \times D^2)$ for some knot K . The first contribution $S^3 \setminus (D^2 \times S^1) = S^1 \times D^2$ can be chosen using a constant embedding, i.e. we have for the integral (4)

$$S_{EH}(\Sigma) = \int_{\Sigma} H \sqrt{h} d^3x = \int_{K \times D^2} H \sqrt{h} d^3x + \text{const.} + \text{boundary} \quad (5)$$

where the contribution of $N(K) = K \times D^2$ reflects the dependence on the topology of the 3-manifold Σ . The integral (4) for two different 3-manifolds differs exactly by this expression. From the topological point of view, we can write alternatively

$$K \times D^2 = K \times D^1 \times D^1 = K \times [0, 1]^2.$$

Finally we end up with the action

$$S_{EH}(K \times D^1 \times D^1) = \int_{K \times D^1 \times D^1} H \sqrt{h} d^3x \quad . \quad (6)$$

Obviously, the complexity of the embedding is given by the knot K or by a plane like $K \times D^1$. Without loss of generality, we choose a product metric and consider the mean curvature H_K for the embedding $K \times D^1$ to state

$$S_{EH}(K \times D^1 \times D^1) = \int_{D^1} d\theta \int_{K \times D^1} H_K \sqrt{h} d^2x \quad . \quad (7)$$

3.2 Weierstrass and spin representation of immersed submanifolds

In this subsection we will describe the theory of immersions using spinors. The theory will be presented stepwise. We start with a toy model of an immersion of a surface into the 3-dimensional Euclidean space. Then we discuss how this map can be extended to an immersion of a 3-manifold into a 4-manifold.

Let $f : M^2 \rightarrow \mathbb{R}^3$ be a smooth map of a Riemannian surface with injective differential $df : TM^2 \rightarrow T\mathbb{R}^3$, i.e. an immersion. In the *Weierstrass representation* one expresses a *conformal minimal* immersion f in terms of a holomorphic function $g \in \Lambda^0$ and a holomorphic 1-form $\mu \in \Lambda^{1,0}$ as the integral

$$f = \text{Re} \left(\int (1 - g^2, i(1 + g^2), 2g)\mu \right) .$$

An immersion of M^2 is conformal if the induced metric g on M^2 has components

$$g_{zz} = 0 = g_{\bar{z}\bar{z}}, \quad g_{z\bar{z}} \neq 0$$

and it is minimal if the surface has minimal volume. Now we consider a spinor bundle S on M^2 (i.e. $TM^2 = S \otimes S$ as complex line bundles) and with the splitting

$$S = S^+ \oplus S^- = \Lambda^0 \oplus \Lambda^{1,0}$$

Therefore the pair (g, μ) can be considered as spinor field φ on M^2 . Then the Cauchy-Riemann equation for g and μ is equivalent to the Dirac equation $D\varphi = 0$. The generalization from a conformal minimal immersion to a conformal immersion was done by many authors (see the references in [Fri98]) to show that the spinor φ now fulfills the Dirac equation

$$D\varphi = H\varphi \quad (8)$$

where H is the mean curvature (i.e. the trace of the second fundamental form). The minimal case is equivalent to the vanishing mean curvature $H = 0$ recovering the equation above. Friedrich [Fri98] uncovered the relation between a spinor Φ on \mathbb{R}^3 and the spinor $\varphi = \Phi|_{M^2}$: if the spinor Φ fulfills the Dirac equation $D\Phi = 0$ then the restriction $\varphi = \Phi|_{M^2}$ fulfills equation (8) and $|\varphi|^2 = \text{const}$. Therefore we obtain

$$H = \bar{\varphi} D\varphi \quad (9)$$

with $|\varphi|^2 = 1$.

After this exercise we are ready to consider the integral (7). Here we have an immersion of $I : S^1 \times D^1 \rightarrow \mathbb{R}^3$ with image the thickened knot $\text{im}(I) = T(K) = K \times D^1$. This immersion I can be defined by a spinor φ on $S^1 \times D^1$ fulfilling the Dirac equation

$$D\varphi = H\varphi \quad (10)$$

with $|\varphi|^2 = 1$ (or an arbitrary constant) (see Theorem 1 of [Fri98]). As discussed above a spinor bundle over a surface splits into two sub-bundles $S = S^+ \oplus S^-$ with the corresponding splitting of the spinor φ in components

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

and we have the Dirac equation

$$D\varphi = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = H \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

with respect to the coordinates (z, \bar{z}) on $S^1 \times D^1$.

In dimension 3, the spinor bundle has the same fiber dimension as the spinor bundle S (but without a splitting $S = S^+ \oplus S^-$ into two sub-bundles). Now we define the extended spinor ϕ over the solid torus $T^3 = S^1 \times D^1 \times D^1 = S^1 \times D^2$ via the restriction $\phi|_{S^1 \times D^1} = \varphi$. The spinor ϕ is constant along the normal vector $\partial_N \phi = 0$ fulfilling the 3-dimensional Dirac equation

$$D^{3D}\phi = \begin{pmatrix} \partial_N & \partial_z \\ \partial_{\bar{z}} & -\partial_N \end{pmatrix} \phi = H\phi \quad (11)$$

induced from the Dirac equation (10) via restriction and where $|\phi|^2 = \text{const}$. Especially one obtains for the mean curvature of the knotted solid torus $K \times D^2$ (up to a constant from $|\phi|^2$)

$$H = \bar{\phi} D^{3D}\phi. \quad (12)$$

3.3 Deformation of the Immersion and the spectrum of the Dirac operator

Now we will discuss the change of the immersion by a diffeomorphism. But first, we will remark that (10) and (11) are eigenvalue equations. The eigenvectors correspond to immersions where the eigenvalue is the mean curvature of this immersion. Then any other immersion corresponds to a linear combination of eigenvectors. The mean curvature of this immersion is also a linear combination of the eigenvalues. In particular, there is also the eigenvector to the eigenvalue 0, called the minimal immersion. Thus, we obtain a quantized (mean) curvature as eigenvalues of a Dirac operator. This approach has some similarities with the spectral triple in noncommutative geometry [Con95]. But in contrast to noncommutative geometry, we start with the simple model to use an exotic smoothness structure. Why did we obtain a similar result? There are many hints that an exotic \mathbb{R}^4 is a noncommutative space in the sense of Connes. We partly worked out this theory using wild embeddings [AMK13].

Now we will discuss the deformation of a immersion using a diffeomorphism. Let $I : \Sigma \hookrightarrow M$ be an immersion of Σ (3-manifold) into M (4-manifold). A deformation of an immersion $I' : \Sigma' \hookrightarrow M'$ are diffeomorphisms $f : M \rightarrow M'$ and $g : \Sigma \rightarrow \Sigma'$ of M and Σ , respectively, so that

$$I \circ f = g \circ I'.$$

One of the diffeomorphism (say f) can be absorbed into the definition of the immersion and we are left with one diffeomorphism $g \in \text{Diff}(\Sigma)$ to define the deformation of the immersion I . But as stated above, the immersion is directly given by an integral over the spinor ϕ on Σ fulfilling the Dirac equation (11). Therefore we have to discuss the action of the diffeomorphism group $\text{Diff}(\Sigma)$ on the Hilbert space of L^2 -spinors fulfilling the Dirac equation. This case was considered in the literature [DD13]. The spinor space $S_{g,\sigma}(\Sigma)$ on Σ depends on two ingredients: a (Riemannian) metric g and a spin structure σ (labeled by the number of elements in $H^1(\Sigma, \mathbb{Z}_2)$). Let us consider the group of orientation-preserving diffeomorphism $\text{Diff}^+(\Sigma)$ acting on g (by pullback f^*g) and on σ (by a suitable defined pullback $f^*\sigma$). The Hilbert space of L^2 -spinors of $S_{g,\sigma}(\Sigma)$ is denoted by $H_{g,\sigma}$. Then according to [DD13], any $f \in \text{Diff}^+(\Sigma)$ leads in exactly two ways to a unitary operator U from $H_{g,\sigma}$ to $H_{f^*g,f^*\sigma}$. The (canonically) defined Dirac operator is equivariant with respect to the action of U and the spectrum is invariant under (orientation-preserving) diffeomorphisms. But by the discussion above, we also do not change the immersion by a diffeomorphism. So, our whole approach is independent on a concrete coordinate system.

3.4 The Dirac action in 3 dimensions and the 4-dimensional Dirac equation

By using the relation (12) above we obtain for the integral (4)

$$\int_{K \times D^2} H_K \sqrt{h} d\theta d^2x = \int_{K \times D^2} \bar{\phi} D^{3D} \phi \sqrt{h} d\theta d^2x \quad (13)$$

i.e. the Dirac action on the knotted solid torus $K \times D^2 = T^3(K)$. But that is not the expected result, we obtain only a 3-dimensional Dirac action leaving us with the question to extend the action to four dimensions.

Let $\iota : T^3 \hookrightarrow M$ be an immersion of the solid torus $\Sigma = T^3$ into the 4-manifold M with the normal vector \mathbf{N} . At this stage one can consider an arbitrary 3-manifold Σ instead of the 3-torus. The spin bundle S_M of the 4-manifold splits into two sub-bundles S_M^\pm where one subbundle, say S_M^+ , can be related to the spin bundle S_Σ of the 3-manifold. Then the spin bundles are related by $S_\Sigma = \iota^* S_M^+$ with the same relation $\phi = \iota_* \Phi$ for the spinors ($\phi \in \Gamma(S_\Sigma)$ and $\Phi \in \Gamma(S_M^+)$). Let $\nabla_X^M, \nabla_X^\Sigma$ be the covariant derivatives in the spin bundles along a vector field X as section of the bundle $T\Sigma$. Then we have the formula

$$\nabla_X^M(\Phi) = \nabla_X^\Sigma \phi - \frac{1}{2}(\nabla_X \mathbf{N}) \cdot \mathbf{N} \cdot \phi \quad (14)$$

with the obvious embedding $\phi \mapsto \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \Phi$ of the spinor spaces. The expression $\nabla_X \mathbf{N}$ is the second fundamental form of the immersion where the trace $\text{tr}(\nabla_X \mathbf{N}) = 2H$ is related to the mean curvature H . Then from (14) one obtains a similar relation between the corresponding Dirac operators

$$D^M \Phi = D^{3D} \phi - H \phi \quad (15)$$

with the Dirac operator D^{3D} defined via (11). Together with equation (11) we obtain

$$D^M \Phi = 0 \quad (16)$$

i.e. Φ is a parallel spinor.

3.5 The extension to the 4-dimensional Dirac action

Above we obtained a relation (15) between a 3-dimensional spinor ϕ on the 3-manifold $\Sigma = D^2 \times S^1$ fulfilling a Dirac equation $D^\Sigma \phi = H \phi$ (determined by the immersion $\Sigma \rightarrow M$ into a 4-manifold M) and a 4-dimensional spinor Φ on a 4-manifold M with fixed chirality ($\in \Gamma(S_M^+)$ or $\in \Gamma(S_M^-)$) fulfilling the Dirac equation $D^M \Phi = 0$. At first we consider the variation

$$\delta \int_{K \times D^2} \bar{\phi} D^{3D} \phi \sqrt{g} d\theta d^2 x = 0 \quad (17)$$

of the 3-dimensional action leading to the Dirac equations

$$D^{3D} \phi = 0 \quad D^{3D} \bar{\phi} = 0 \quad (18)$$

or to

$$H = 0,$$

a characterization of the immersion $K \times D^2$ of the solid torus $D^2 \times S^1$ with minimal mean curvature. This variation can be understood as a variation of the

(conformal) immersion. In contrast, the extension of the spinor ϕ (as solution of (18)) to the 4-dimensional spinor Φ by using the embedding

$$\Phi = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad (19)$$

can be only seen as immersion, if (and only if) the 4-dimensional Dirac equation

$$D^M \Phi = 0 \quad (20)$$

on M is fulfilled (using relation (15)). This Dirac equation is obtained by varying the action

$$\delta \int_M \bar{\Phi} D^M \Phi \sqrt{g} d^4x = 0 \quad (21)$$

Importantly, this variation has a different interpretation in contrast to varying the 3-dimensional action. Both variations look very similar. But in (21) we vary over smooth maps $\Sigma = D^2 \times S^1 \rightarrow M$ which are not conformal immersions (i.e. represented by spinors Φ with $D^M \Phi \neq 0$). Only the choice of the extremal action selects the conformal immersion among other smooth maps. Especially the spinor Φ (as solution of the 4-dimensional Dirac equation) is localized at the immersed 3-manifold Σ (with respect to the embedding (19)). The 3-manifold Σ moves along the normal vector (see the relation (14) between the covariant derivatives representing a parallel transport).

3.6 Matter as knot complements

In the previous subsections we presented a formalism to describe the immersion of a solid torus $D^2 \times S^1$ with a knotted solid torus $D^2 \times K$ as image. Now we will go back to our original view (see subsection 3.1). There we considered the 3-manifold $\Sigma = (S^3 \setminus (D^2 \times S^1)) \cup (K \times D^2)$ which is equally given by $\Sigma = (S^3 \setminus (D^2 \times K)) \cup (D^2 \times S^1)$. Then the spinor ϕ on $D^2 \times K$ is related to the spinor ϕ' on $S^3 \setminus (D^2 \times K)$ by a constant, which is the normalization of the spinor Φ on Σ with $\Phi|_{D^2 \times K} = \phi$. But then the spinors ϕ and ϕ' fulfill the same dynamics, the Dirac equation. But what does it mean? From the view point of quantum mechanics, the spinor ϕ as immersion of $D^2 \times S^1$ is non-zero on the space of possible positions. If we make the obvious assumption that the complement of this space $D^2 \times S^1$ is the particle (represented by the spinor) then the particle must be the complement $S^3 \setminus (D^2 \times K)$ of the knotted solid torus. This space is also called the knot complement. A knot complement is a compact 3-manifold with boundary a torus T^2 . After the extension to the 4-manifold M , the spinor Φ represents the dynamics of the knot complement in the 4-manifold. Finally we state:

Matter is represented by complements $S^3 \setminus (D^2 \times K)$ of knots K with a dynamics determined by the Dirac equation (20).

Currently this statement is not a large restriction. There are infinitely many knots and we do not know which knot represents the electron or neutrino. But

for knot complements, there is a simple division into two classes: knots with a knot complement admitting a homogenous, hyperbolic metric (a metric of constant negative curvature in every direction) and knots not admitting such a metric. In [AMR12], we discussed the non-hyperbolic case and showed that the corresponding 3-manifolds are representing the interaction. Therefore we are left with hyperbolic knot complements. In the next section we will show that these knot complements have the right properties to describe fermions.

4 The physical interpretation

In this section we will discuss the physical interpretation of the mathematical results above including the limits of this approach. In particular we will prove the conjecture that hyperbolic knot complements, i.e. 3-manifolds $S^3 \setminus (D^2 \times K)$ admitting a homogenous, hyperbolic metric, representing the fermions. We used the spinor representation to express the immersion of the submanifold. Here we will further clarify the following questions: Does the submanifold (the knot complement) has the properties of a spinor fulfilling the Dirac equation? Has it also the properties of matter like non-contractability (state equation $p = 0$)? From a physical point of view, we have to check that the submanifold (=knot complement) has

1. spin $\frac{1}{2}$ (with an appropriated definition),
2. the Dirac equation as equation of motion and
3. the state equation $p = 0$ (non-contractable matter) in the cosmological context.

ad 1. We start with the spin. Our definition is inspired by the work of Friedman and Sorkin [FS80], for the details we refer to the Appendix B. Now we will look for a rotation $R(\theta)$ (rotation w.r.t. an angle θ) which acts on the 4-dimensional spinor Φ . Because of the embedding (19), it is enough to consider the action on the 3-dimensional spinor ϕ . Then a rotation as element of $SO(3)$ must be represented by a diffeomorphism, i.e. we have the representation $R : SO(3) \rightarrow Diff(\Sigma)$ where $R(\theta)$ is a one-parameter subgroup of diffeomorphisms. We call ϕ a spinor if

$$\phi \circ R(2\pi)^* = -\phi \quad \text{or} \quad R(2\pi) = -1$$

in the notation of Appendix B. From the topological point of view, this rotation is located in the component of the diffeomorphism group which is not connected to the identity. The existence of these rotations is connected to the complexity of the 3-manifold. As shown by Hendriks [Hen77], these rotations do **not** exist in sums of 3-manifolds containing

- $\mathbb{R}P^2 \times S^1$ with the Klein bottle $\mathbb{R}P^2$
- S^2 fiber bundle over S^1 and

- for 3-manifolds with finite fundamental group having a cyclic 2-Sylow subgroup⁴.

In case of hyperbolic 3-manifolds (the knot complements) one has an infinite fundamental group and therefore it has spin $\frac{1}{2}$.

ad 2. This part was already shown. Using the variation (21) we obtain the 4-dimensional Dirac equation (20) in case of an immersion. Then the spinor is directly interpretable as the immersion, see subsection 3.2.

ad 3. In cosmology, one has to introduce a state equation

$$p = w \cdot \rho$$

between the pressure and the energy density. Matter as formed by fermions is characterized by the state equation $p = 0$ or $w = 0$. Equivalently, matter is incompressible and the energy density $\rho \sim a^{-3}$ scales like the inverse volume of the 3-space w.r.t. scaling factor a . The hyperbolic 3-manifold H , i.e. the complement of the hyperbolic knot, has a torus boundary $T^2 = \partial H$, i.e. H admits a hyperbolic structure in the interior only. It should also have the property of incompressibility. But what does it mean? As a model we consider the following 3-manifold

$$N = H \cup_{T^2} G$$

where the two manifolds H and G have a common boundary, the torus. H represents the matter (by our assumption) and G is the surrounding space, i.e. we take N as a model for the cosmos. Furthermore we assume that N scales w.r.t. the scaling factor a , i.e. $\text{vol}(N) \sim a^3$. The energy density is the total energy E_H of the matter per volume or

$$\rho = \frac{E_H}{\text{vol}(N)} \quad .$$

The total energy E_H is related to the scalar curvature, see appendix C. Using (25), we obtain for the total energy of the hyperbolic 3-manifold H the total energy E_H with

$$E_H = \text{vol}(H) \cdot \left(\frac{1}{\kappa} R_H + \rho_c \right) \quad .$$

Therefore we will get the scaling law $\rho \sim a^{-3}$ only for $E_H \sim a^0$ by using $\text{vol}(N) \sim a^3$. It is an amazing fact that the properties of hyperbolic 3-manifolds agree with this demand. One property of hyperbolic 3-manifolds is central: *Mostow rigidity*. As shown by Mostow [Mos68], every hyperbolic n -manifold $n > 2$ of finite volume has the property: *Every diffeomorphism (especially every conformal transformation) of a hyperbolic n -manifold with finite volume is induced by an isometry*. Therefore one cannot scale a finite-volume, hyperbolic 3-manifold. Then the volume $\text{vol}()$ and the curvature are topological invariants. But then E_H is also a topological invariant with the scaling behaviour $E_H \sim a^0$

⁴ A 2-Sylow subgroup of a finite group (here the fundamental group) is a subgroup whose order is a power of 2 (possibly 2^0) and which is properly contained in no larger Sylow subgroup. We note that all 2-Sylow subgroups of a given group are isomorphic.

of a topological invariant. Finally we obtain the scaling of matter in cosmology to be a^{-3} or $w = 0$.

Finally: *Fermions are represented by hyperbolic knot complements.*

5 The Brans conjecture: generating sources of gravity

We only do direct geometric observations within some local, human-scaled coordinate patch, including, of course, interpolations of signals received from sources outside this patch. From this, we usually assume that spacetime has the simplest global smoothness structure. Suppose it does not, so that spacetime is exotically smooth. For example, suppose we observe only a single mass outside our local region and it looks like a black hole. Normally, we assume we can extrapolate data arriving in our standard coordinate patch on earth all the way back to the vicinity of the black hole. We ask: "what if the smoothness structure does not allow this?"

This question is at the core of the Brans conjecture. Exotic spacetimes like the exotic \mathbb{R}^4 have the property that there is no foliation like $\mathbb{R}^3 \times \mathbb{R}$ otherwise the spacetime has a standard smoothness structure. But all other foliations break the strong causality, i.e. there is no unique geodesics going in the future or past (see the discussion in [AMR12]). In this paper we will go a step further and will interpret the deviation of the smoothness structure from the standard smoothness structure as sources of gravity. In particular we will use the theory above to identify the sources as fermions.

5.1 Large exotic \mathbb{R}^4

At first we will discuss the case of a large exotic \mathbb{R}^4 as described in subsection 2.2. Starting point for the construction is a topologically slice but smoothly non-slice knot K (like the pretzel knot $(-3, 5, 7)$ in Fig. 2) in D^4 . Let X_K be the two-handlebody obtained by attaching a two-handle to D^4 along K with framing 0. Then the open 4-manifold

$$R^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \quad (22)$$

where $\text{int}\rho(X_K)$ is the interior of $\rho(X_K)$, is homeomorphic but non-diffeomorphic to \mathbb{R}^4 with the standard smoothness structure (both pieces are glued along the common boundary ∂X_K). The boundary ∂X_K can be constructed by a 0-framed surgery along K , i.e. $\partial X_K = (S^3 \setminus (K \times D^2)) \cup_{T^2} D^2 \times S^1$ glued along the torus respecting the framing. For the Einstein-Hilbert action we obtain

$$S_{EH}(R^4) = \int_{\mathbb{R}^4 \setminus \text{int}\rho(X_K)} R\sqrt{g}d^4x + \int_{X_K} R\sqrt{g}d^4x + \int_{\partial X_K} H\sqrt{h}d^3x \quad (23)$$

where H is the mean curvature (trace of the second fundamental form) w.r.t. the metric $h = g|_{\partial X_K}$. One word about the boundary term. Usually one obtains

two boundary terms but with a different sign. The cancellation of these terms uses implicitly the fact that the boundary (the 3-manifold) and orientation-reversing boundary are related by an orientation-reversing diffeomorphism so that both boundary terms cancel. But for most 3-manifolds among them the hyperbolic 3-manifolds it fails, i.e. there is no orientation-reversing diffeomorphism and the two boundary contributions are different. The boundary ∂X_K (for the pretzel knot) is also a hyperbolic 3-manifold with no orientation-reversing diffeomorphism. Therefore we obtain a contribution from the boundary in the action (23). By the formalism above, we are able to construct the Dirac action on ∂X_K

$$\int_{\partial X_K} H \sqrt{h} d^3 x = \int_{\partial X_K} \bar{\phi} D^{3D} \phi \sqrt{h} d^3 x$$

and extend them

$$S_{EH}(R^4) = \int_{\mathbb{R}^4 \setminus \text{int} \rho(X_K)} R \sqrt{g} d^4 x + \int_{X_K} R \sqrt{g} d^4 x + \int_{R^4} \bar{\Phi} D \Phi \sqrt{g} d^4 x$$

to the whole 4-manifold (but at least to $\partial X_K \times [0, 1]$). Then we can simplify the action to

$$S_{EH}(R^4) = \int_{R^4} R \sqrt{g} d^4 x + \int_{R^4} \bar{\Phi} D \Phi \sqrt{g} d^4 x$$

where the spinor is concentrated around $\partial X_K \times [0, 1]$. Finally we obtain the (chiral, see the embdding (19)) fermion field Φ as source term which is directly related to the exotic smoothness structure.

5.2 Small exotic \mathbb{R}^4

In case of a small exotic \mathbb{R}^4

$$\mathbb{R}_\theta^4 = \text{int} (A_{\text{cor}k} \cup_N W \cup_N W \cup_N \dots)$$

we have a different decomposition (see [BG96] for an explicit handle decomposition) using the machinery of Casson handles. But the main results remain the same, i.e. we end up with the action

$$S_{EH}(R^4) = \int_{\mathbb{R}_\theta^4} R \sqrt{g} d^4 x + \int_{\mathbb{R}_\theta^4} \bar{\Phi} D \Phi \sqrt{g} d^4 x$$

but with an important difference. The spinor Φ is concentrated along the boundary regions $N \times [0, 1]$ like in the previous case but now the underlying structure of the decomposition is a tree (the tree of the Casson handle). From the physical point of view, we obtain the creation of spinors if we go along this tree.

6 Conclusion

In this paper we confirmed the Brans conjecture in the form that exotic smoothness is a generator of sources in gravity. As example we choose the exotic \mathbb{R}^4 but the proof is general enough to include also all other cases. The compact case was confirmed in [AMR12]. As a technical tool we used the spin representation of immersed surfaces to describe fermions as knot complements. It is interesting that fermions are created naturally in both families (large and small) of exotic \mathbb{R}^4 's. By using more complicated knots, one can also describe the interaction between the fermions (see [AMR12] again). These connecting pieces are so-called torus bundles (remember the boundary of the knot complement is a torus). There are three types of torus bundles and we related them to the known gauge theories. In our forthcoming work, we will describe this relation more fully. Secondly we have done a lot of work to show a relation to quantum gravity.

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A Connected and boundary-connected sum of manifolds

Now we will define the connected sum $\#$ and the boundary connected sum \natural of manifolds. Let M, N be two n -manifolds with boundaries $\partial M, \partial N$. The *connected sum* $M \# N$ is the procedure of cutting out a disk D^n from the interior $\text{int}(M) \setminus D^n$ and $\text{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \sqcup \partial M$ and $S^{n-1} \sqcup \partial N$, respectively, and gluing them together along the common boundary component S^{n-1} . The boundary $\partial(M \# N) = \partial M \sqcup \partial N$ is the disjoint sum of the boundaries $\partial M, \partial N$. The *boundary connected sum* $M \natural N$ is the procedure of cutting out a disk D^{n-1} from the boundary $\partial M \setminus D^{n-1}$ and $\partial N \setminus D^{n-1}$ and gluing them together along S^{n-2} of the boundary. Then the boundary of this sum $M \natural N$ is the connected sum $\partial(M \natural N) = \partial M \# \partial N$ of the boundaries $\partial M, \partial N$.

B Spin $\frac{1}{2}$ from space a la Friedman and Sorkin

As shown by Friedman and Sorkin [FS80], the calculation of the angular momentum in the ADM formalism is connected to special diffeomorphisms $R(\theta)$ (rotation parallel to the boundary w.r.t. the angle θ). So, one can speak of spin $\frac{1}{2}$, in case of $R(2\pi) \neq -1$. Interestingly, all hyperbolic 3-manifolds having these diffeomorphisms.

In the following we made use of the work [FS80] in the definition of the angular momentum in ADM formalism. In this formalism, one has the 3-manifold Σ together with a time-like foliation of the 4-manifold $\Sigma \times \mathbb{R}$. For simplicity, we consider the interior of the 3-manifold or we assume a 3-manifold without boundary. The configuration space \mathcal{M} in the ADM formalism is the space of all Riemannian metrics of Σ modulo diffeomorphisms. On this space we define the linear functional $\psi : \mathcal{M} \rightarrow \mathbb{C}$ calling it a state. In case of a

many-component object like a spinor one has the state $\psi : \mathcal{M} \rightarrow \mathbb{C}^n$. Let g_{ab} be a metric on Σ and we define the generalized position operator

$$\hat{g}_{ab}\psi(g) = g_{ab}\psi(g)$$

together with the conjugated momentum

$$\hat{\pi}^{ab}\psi(g) = -i\frac{\delta}{\delta g_{ab}}\psi(g) \quad .$$

Let ϕ_α with $\alpha = 1, 2, 3$ be vector fields fulfilling the commutator rules $[\phi_\alpha, \phi_\beta] = -\epsilon_{\alpha\beta\gamma}\phi_\gamma$ generating an isometric realization of the $SO(3)$ group on the 3-manifold Σ . The angular momentum corresponding to the initial point (g_{ab}, π^{ab}) with the conjugated momentum $\pi^{ab} = (16\pi)^{-1}(-K^{ab} + g^{ab}K)\sqrt{g}$ (in the ADM formalism) and the extrinsic curvature K_{ab} is given by

$$J_\alpha = - \int_{\Sigma} \mathcal{L}_{\phi_\alpha}(g_{ab})\pi^{ab} d^3x$$

with the Lie derivative $\mathcal{L}_{\phi_\alpha}$ along ϕ_α . The action of the corresponding operator \hat{J}_α on the state $\psi(g)$ can be calculated to be

$$\hat{J}_\alpha\psi(g) = -i\frac{d}{d\theta}\psi \circ R_\alpha(\theta)^*(g)|_{\theta=0}$$

where $R_\alpha(\theta)$ is a 1-parameter subgroup of diffeomorphisms generated by ϕ_α . Then a rotation will be generated by

$$\exp(2\pi i \hat{J})\psi = \psi \circ R(2\pi)^* \quad .$$

Now a state ψ carries spin $\frac{1}{2}$ iff $\psi \circ R(2\pi)^* = -\psi$ or $R(2\pi) = -1$. In this case the diffeomorphism $R(2\pi)$ is not located in the component of the diffeomorphism group which is connected to the identity (or equally it is not generated by coordinate transformations).

C Scalar curvature and energy density

Let us consider a Friedmann-Robertson-Walker-metric

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k$$

on $N \times [0, 1]$ with metric h_{ik} on N and the Friedmann equation

$$\left(\frac{\dot{a}(t)}{c \cdot a(t)}\right)^2 + \frac{k}{a(t)^2} = \kappa \frac{\rho}{3}$$

with the scaling factor $a(t)$, curvature $k = 0, \pm 1$ and $\kappa = \frac{8\pi G}{c^2}$. As an example we consider a 3-dimensional submanifold N with energy density ρ_N and curvature R_N (related to h) fixed embedded in the spacetime. Next we assume that the 3-manifold N posses a homogenous metric of constant curvature. For a fixed time t , the scalar curvature of N is proportional to

$$R_N \sim \frac{3k}{a(t)^2}$$

and by using the Friedmann equation above, one obtains

$$\rho_N = \frac{1}{\kappa} R_N + \rho_c$$

with the critical density

$$\rho_c = \frac{3}{\kappa} \left(\frac{\dot{a}(t)}{c \cdot a(t)}\right)^2 = \frac{3H^2}{\kappa}$$

and the Hubble constant H

$$H = \frac{\dot{a}}{c \cdot a} \quad .$$

The total energy of N is given by

$$E_N = \int_N \rho_N \sqrt{h} d^3x = \frac{1}{\kappa} \int_N R_N \sqrt{h} d^3x + \rho_c \cdot \text{vol}(N) \quad . \quad (24)$$

For a space with constant curvature R_N we obtain

$$E_N = \left(\frac{1}{\kappa} R_N + \rho_c \right) \cdot \text{vol}(N) \quad (25)$$

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